Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 2, pp. 376–385, 2001

DESIGN OF PLANE THERMOELASTIC COMPOSITE CONSTRUCTIONS WITH UNIFORMLY STRESSED REINFORCEMENT

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UDC 539.4

The problem of uniformly stressed reinforcement of plane composite constructions under thermoforce loading is formulated. An asymptotic analysis of the corresponding boundary-value problem is performed. Based on this analysis, it is shown that the problem may have two solutions due to the significant nonlinearity of static boundary conditions. An iterative method for solving the problem is proposed. Particular analytical and numerical solutions are analyzed, and the level of influence of the thermal action on uniformly stressed reinforced constructions is studied.

One criterion of rational design of fibrous composite constructions under static loading is the equal stress of fibers along their trajectories, since the bearing capacity of high-modulus and high-strength reinforcement is used most completely if this criterion is satisfied, and the binder is responsible only for uniform redistribution of loads on elementary fibers. The problem of rational reinforcement (RR) of plane composite constructions by uniformly stressed high-modulus constant-section fibers, which takes into account the thermal action, is formulated in [1]. Nevertheless, the qualitative and quantitative effect of temperature on the RR structure has not yet been adequately studied. In the present paper, an asymptotic analysis of the system of resolving equations of the thermoelastic RR problem is performed, and the effect of the thermal action on the reinforcement structure and on the stress–strained state of the construction with uniformly stressed fibers is studied.

1. System of Resolving Equations and Boundary Conditions. A complete closed normalized system of resolving equations of the RR problem, which describes the behavior of plane thermoelastic constructions that are statically loaded and reinforced by two families of uniformly stressed fibers (the binder and fiber materials are assumed to be isotropic, and their behavior is assumed to be linearly elastic), includes the equations of equilibrium

$$A_i(\boldsymbol{\omega},\boldsymbol{\alpha}) + \lambda B_i(\boldsymbol{\omega},\boldsymbol{u},\theta) = -b_i(\boldsymbol{\omega}) \equiv -\Big(aF_{ci} + \sum_k \omega_k F_{ki}\Big) \qquad (i = 1,\ 2) \tag{1.1}$$

written in displacements [1], the conditions of constant cross sections of the fibers [1]

$$\partial_s(\alpha_k,\omega_k) + \omega_k \partial_n(\alpha_k,\alpha_k) = 0 \qquad (k = 1,\ 2), \tag{1.2}$$

the conditions of uniformly stressed reinforcement [1]

$$\partial_s(\alpha_k,u_1)\cos\alpha_k + \partial_s(\alpha_k,u_2)\sin\alpha_k - \alpha_{ak}\theta = \varepsilon_k = \sigma_k/E_k = \text{const} \quad (k = 1,\ 2), \tag{1.3}$$

and the equation of plane stationary thermal conductivity [2]

Institute of Theoretical and Applied Mechanics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 2, pp. 213–223, March–April, 2001. Original article submitted May 31, 2000.

 0021-8944/01/4202-0376 $25.00

$$(\Lambda_{11}\theta_{,1} + \Lambda_{12}\theta_{,2})_{,1} + (\Lambda_{21}\theta_{,1} + \Lambda_{22}\theta_{,2})_{,2} + \mu(\theta_* - \theta) = -Q(\boldsymbol{\omega}) \equiv -\Big(aQ_c + \sum_k \omega_k Q_k\Big). \tag{1.4}$$

Here A_i, B_i, ∂_s, and ∂_n are differential operators of the form

$$A_i(\boldsymbol{\omega}, \boldsymbol{\alpha}) = (-1)^i \sum_k \sigma_k \omega_k l_{kj} \partial_s(\alpha_k, \alpha_k),$$

$$B_i(\boldsymbol{\omega}, \boldsymbol{u}, \theta) = aa_1[u_{i,ii} + \nu u_{j,ij} + (1-\nu)(u_{j,ij} + u_{i,jj})/2 - \alpha_c\theta_{,i}/a_2] \tag{1.5}$$

$$- \sum_k [a_1(u_{i,i} + \nu u_{j,j} - \alpha_c\theta/a_2)\omega_{k,i} + a_2(u_{j,i} + u_{i,j})\omega_{k,j}/2] \quad (j = 3-i, \quad i = 1,\ 2);$$

$$\partial_s(\alpha_k, f) = f_{,1}\cos\alpha_k + f_{,2}\sin\alpha_k, \quad \partial_n(\alpha_k, f) = -f_{,1}\sin\alpha_k + f_{,2}\cos\alpha_k \tag{1.6}$$

(f is an arbitrarily differentiable function),

$$\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}, \qquad \boldsymbol{\omega} = \{\omega_1, \omega_2\}, \qquad \boldsymbol{u} = \{u_1, u_2\}, \tag{1.7}$$

$$\Lambda_{ij} = \Omega^{-1} \sum_k \omega_k \{[\Omega(\lambda_k - \lambda_c) + \lambda_c] l_{ki} l_{kj} + (-1)^{i+j} l_{ks} l_{kr} \lambda_k \lambda_c [\Omega(\lambda_c - \lambda_k) + \lambda_k]^{-1}\}, \tag{1.8}$$

$$(s = 3-i, \qquad r = 3-j, \qquad i,\ j = 1,\ 2).$$

On one part of the contour Γ_p, it is possible to set the static boundary conditions [1]

$$C_n(\boldsymbol{\alpha}, \boldsymbol{\omega}) + \lambda D_n(\boldsymbol{\omega}, \boldsymbol{u}, \theta) = p_n, \qquad C_\tau(\boldsymbol{\alpha}, \boldsymbol{\omega}) + \lambda D_\tau(\boldsymbol{\omega}, \boldsymbol{u}) = 2p_\tau, \tag{1.9}$$

on the other part Γ_u, one can set the kinematic conditions

$$u_i(\Gamma_u) = u_{i0} \qquad (i = 1,\ 2), \tag{1.10}$$

and on the entire contour Γ, it is possible to set the thermal conditions

$$\chi_0[(\Lambda_{11}\theta_{,1} + \Lambda_{12}\theta_{,2})n_1 + (\Lambda_{21}\theta_{,1} + \Lambda_{22}\theta_{,2})n_2 + q_0] + \chi_1(\theta - \theta_0) = 0, \tag{1.11}$$

where

$$C_n(\boldsymbol{\alpha}, \boldsymbol{\omega}) = \sum_k \sigma_k \omega_k \cos^2(\alpha_k - \beta), \qquad C_\tau(\boldsymbol{\alpha}, \boldsymbol{\omega}) = \sum_k \sigma_k \omega_k \sin 2(\alpha_k - \beta), \tag{1.12}$$

$$D_n(\boldsymbol{\omega}, \boldsymbol{u}, \theta) = aa_1[(u_{1,1} + \nu u_{2,2})n_1^2 + (u_{2,2} + \nu u_{1,1})n_2^2 + (1-\nu)(u_{1,2} + u_{2,1})n_1 n_2 - \alpha_c\theta/a_2],$$

$$D_\tau(\boldsymbol{\omega}, \boldsymbol{u}) = aa_2[2(u_{2,2} - u_{1,1})n_1 n_2 + (u_{1,2} + u_{2,1})(n_1^2 - n_2^2)]; \quad n_1 = \cos\beta, \quad n_2 = \sin\beta.$$

In addition to the boundary conditions (1.9)–(1.11), it is necessary to set the boundary conditions for reinforcement intensity on the part of the contour Γ_ω, where the fibers enter the construction [3]:

$$\omega_k(\Gamma_\omega) = \omega_{0k} \qquad (k = 1,\ 2). \tag{1.13}$$

Relations (1.1)–(1.13) are written in the following dimensionless variables: $\sigma_k = \bar\sigma_k/|\bar\sigma_1|$, $\varepsilon_k = \bar\varepsilon_k/|\bar\varepsilon_1|$, $E_k = \bar E_k/\bar E_1$, $u_i = \bar u_i/|D\bar\varepsilon_1|$, $b_i = D\bar b_i/|\bar\sigma_1|$, $F_{ci} = D\bar F_{ci}/|\bar\sigma_1|$, $F_{ki} = D\bar F_{ki}/|\bar\sigma_1|$, $\alpha_{ak} = \bar\alpha_{ak}/\bar\alpha_c$, $\alpha_c = \bar\alpha_c/\bar\alpha_c = 1$, $\lambda_k = \bar\lambda_k/\bar\lambda_c$, $\lambda_c = \bar\lambda_c/\bar\lambda_c = 1$, $\theta = \bar\alpha_c\bar\theta/|\bar\varepsilon_1|$, $\theta_0 = \bar\alpha_c\bar\theta_0/|\bar\varepsilon_1|$, $\theta_* = \bar\alpha_c\bar\theta_*/|\bar\varepsilon_1|$, $Q = \bar\alpha_c D^2\bar Q/|\bar\lambda_c\bar\varepsilon_1|$, $Q_k = \bar\alpha_c D^2\bar Q_k/|\bar\lambda_c\bar\varepsilon_1|$, $Q_c = \bar\alpha_c D^2\bar Q_c/|\bar\lambda_c\bar\varepsilon_1|$, $q_0 = \bar\alpha_c D\bar q_0/|\bar\lambda_c\bar\varepsilon_1|$, $\mu = 2D^2\bar\mu/(h\bar\lambda_c)$, $p_n = \bar p_n/|\bar\sigma_1|$, $p_\tau = \bar p_\tau/|\bar\sigma_1|$, $u_{i0} = \bar u_{i0}/|D\bar\varepsilon_1|$, $l_{k1} = \cos\alpha_k$, $l_{k2} = \sin\alpha_k$, $a = 1-\Omega$, $\Omega = \sum_k \omega_k$, $a_1 = 1/(1-\nu^2)$, and $a_2 = 1/(1+\nu)$ $(i,\ k = 1,\ 2)$. Here $\lambda = \bar E/\bar E_1$ is a small parameter, $\bar\sigma_k$ and $\bar\varepsilon_k$ are the stress and mechanical strain in reinforcement of the kth family, ν is the Poisson's ratio of the binder, $\bar E$ and $\bar E_k$ are and the elasticity moduli of the binder and reinforcement of the kth family, respectively, ω_k and α_k are the intensity and the angle (counted from the direction x_1) of reinforcement by a fiber of the kth family, $\bar u_i$ and $\bar b_i$ are the components of displacement and reduced volume load in the directions $\bar x_i$ of a rectangular Cartesian coordinate system ($\bar x_i = Dx_i$, where $i = 1$ and 2), $\bar F_{ci}$ and $\bar F_{ki}$ are the components of specific volume loads acting on the binder and reinforcement of the kth family, respectively, D and h are the characteristic size and thickness of the plate, $\bar{\alpha}_c$ and $\bar{\alpha}_{ak}$ are the coefficients of linear thermal expansion of the binder and reinforcement of the kth family, $\bar{\lambda}_c$ and $\bar{\lambda}_k$ are the thermal conductivities of the binder and reinforcement of the kth family, $\bar{\theta}$ is the plate-temperature difference in the working and initial states, $\bar{\theta}_0$ is the temperature difference of the plate contour in the working and initial states, $\bar{\theta}_*$ is the temperature difference between the ambient medium (on the side of the front surfaces of the plate) and the initial state of the plate, $\bar{Q}$ is the reduced density of internal heat sources in the fibrous composite, $\bar{Q}_c$ and $\bar{Q}_k$ are the densities of internal heat sources in the binder and reinforcement of the kth family, $\bar{\mu}$ is the coefficient of convective heat exchange between the binder and the ambient medium on the front surfaces of the plate, $\bar{q}_0$ is the heat flux through the side surface of the construction (through the plate edge), β is the angle that defines the direction of the external normal to the contour Γ, $\bar{p}_n$ and $\bar{p}_\tau$ are the normal and tangential stresses on Γ_p, $\bar{u}_{i0}$ is the displacement on the contour, χ_0 and χ_1 are the functions depending on the form of the thermal boundary conditions on the contour, and ω_{0k} is the intensity of reinforcement by a fiber of the kth family, which is set on Γ_ω; summation from 1 to 2 is performed over the index k; the subscript after the comma indicates partial differentiation with respect to the corresponding variable x_i; the unknown functions are α_k, ω_k, u_i, and θ. The functions ω_k should satisfy the conditions

$$\omega_k > 0 \qquad (k = 1,\ 2), \qquad \Omega = \sum_k \omega_k < 1. \tag{1.14}$$

It is shown in [1] that the system of resolving equations (1.1)–(1.4) is a quasilinear system of the mixed-composite type [4], which is closed relative to the unknown functions α_k, ω_k, u_i, and θ ($k,\ i = 1, 2$) and has two complex characteristics generated by the heat-conduction equation (1.4) and two real characteristics that coincide with the trajectories of uniformly stressed fibers.

2. Asymptotic Analysis of the System of Resolving Equations and Boundary Conditions. We turn to zero the small parameter λ in system (1.1) and boundary conditions (1.9). Then, the equations of asymptotic analysis acquire the following form:

$$A_i(\boldsymbol{\omega}, \boldsymbol{\alpha}) = -b_i(\boldsymbol{\omega}), \qquad i = 1,\ 2 \qquad (\lambda \to 0); \tag{2.1}$$

$$\partial_s(\alpha_k, \omega_k) + \omega_k \partial_n(\alpha_k, \alpha_k) = 0, \qquad k = 1,\ 2; \tag{2.2}$$

$$(\Lambda_{11}\theta_{,1} + \Lambda_{12}\theta_{,2})_{,1} + (\Lambda_{21}\theta_{,1} + \Lambda_{22}\theta_{,2})_{,2} + \mu(\theta_* - \theta) = -Q(\boldsymbol{\omega}); \tag{2.3}$$

$$\partial_s(\alpha_k, u_1)\cos\alpha_k + \partial_s(\alpha_k, u_2)\sin\alpha_k = \varepsilon_k + \alpha_{ak}\theta, \qquad k = 1,\ 2. \tag{2.4}$$

The boundary conditions (1.9)–(1.11) and (1.13) for $\lambda \to 0$ are reduced to the form

$$C_n(\boldsymbol{\alpha}, \boldsymbol{\omega}) = p_n, \qquad C_\tau(\boldsymbol{\alpha}, \boldsymbol{\omega}) = 2p_\tau, \qquad (x_1, x_2) \in \Gamma_p; \tag{2.5}$$

$$u_i(\Gamma_u) = u_{i0}, \qquad i = 1,\ 2; \tag{2.6}$$

$$\chi_0[(\Lambda_{11}\theta_{,1} + \Lambda_{12}\theta_{,2})n_1 + (\Lambda_{21}\theta_{,1} + \Lambda_{22}\theta_{,2})n_2 + q_0] + \chi_1(\theta - \theta_0) = 0; \tag{2.7}$$

$$\omega_k(\Gamma_\omega) = \omega_{0k}, \qquad k = 1,\ 2. \tag{2.8}$$

[Obviously, system (2.1)–(2.4) and boundary conditions (2.5)–(2.8) describe a thermoelastic RR problem with the use of the "fibrous" model of the reinforced layer.]

An analysis of system (2.1)–(2.4) shows that the equations of asymptotic analysis, in contrast to the initial system (1.1)–(1.4), are divided into three closed subsystems: the first subsystem (2.1), (2.2) consists of four quasilinear equations and is closed relative to α_k and ω_k; the second subsystem (2.3) includes one equation and is closed relative to θ for α_k and ω_k known from (2.1) and (2.2); the third subsystem (2.4) consists of two equations and is closed relative to u_k ($k = 1, 2$) for α_k and θ known from (2.1)–(2.3). The boundary conditions (2.5)–(2.8) are divided in a similar manner: conditions (2.5) and (2.8) include four equalities and are closed (for $\Gamma_p = \Gamma_\omega$) relative to the boundary values of the functions α_k and ω_k; the thermal conditions (2.7) for α_k and ω_k known from the solution of the boundary-value problem (2.1), (2.2), (2.5), (2.8) define the boundary values of the function θ or the heat-flux conditions on the contour Γ, and the kinematic conditions (2.6) define the values of two functions u_k ($k = 1, 2$) on the contour Γ_u. Hence, the use of asymptotic analysis allows us to divide the previously connected problems of determining the RR parameters, the temperature field, and the stress–strained state in the construction into a number of subproblems, which may be sequentially integrated.

It is shown [5] that subsystem (2.1), (2.2) has two double real characteristics determined by the angles α_k (i.e., coinciding with reinforcement trajectories). However, this subsystem cannot be brought to the characteristic form [6]; therefore, it refers neither to the hyperbolic nor to the parabolic type [in particular, system (2.1), (2.2) degenerates into a parabolic system for $\alpha_1 = \alpha_2$]. If volume loads are not considered in the RR problem, i.e., $b_i = 0$ in (2.1), then system (2.1), (2.2) is reducible, and its general integral has the following form:

$$-x_2 \cos\alpha_k + x_1 \sin\alpha_k = f_k(\alpha_k),$$
$$\omega_k = g_k(\alpha_k)[x_2 \sin\alpha_k + x_1 \cos\alpha_k - f_k'(\alpha_k)]^{-1} \qquad (k = 1,\ 2). \tag{2.9}$$

Here f_k and g_k are arbitrary functions of one argument; the prime indicates the derivative with respect to this argument. If the values of the functions α_k and ω_k are known on a certain line (in particular, on the contour of the construction), then the functions f_k and g_k ($k = 1, 2$) may be determined on this line using Eq. (2.9). Let, for instance, on the contour Γ_ω, where the fibers enter the construction, the values of $\alpha_k(\Gamma_\omega) = \alpha_{0k}(s)$ and $\omega_k(\Gamma_\omega) = \omega_{0k}(s)$ be known [see (1.13) and (2.8)]; then from (2.9) we obtain

$$f_k(\alpha_k) = -\xi_2(s)\cos\alpha_{0k}(s) + \xi_1(s)\alpha_{0k}(s),$$
$$g_k(\alpha_k) = \omega_{0k}(s)[-\xi_2'(s)\cos\alpha_{0k}(s) + \xi_1'(s)\sin\alpha_{0k}(s)]/\alpha_{0k}'(s), \qquad k = 1,\ 2, \tag{2.10}$$

where ξ_i are the functions defining the equation of the contour Γ_ω [$x_i = \xi_i(s)$, where $i = 1$ and 2].

Let us show that the values of the functions α_k on the contour may be determined from the static boundary conditions (2.5). Let the fibers enter the construction on a part of the contour, where the static boundary conditions are set (Γ_p coincides with Γ_ω), and go out on a part of the contour Γ_u, where the kinematic conditions are set. Then, on the contour Γ_p, we know the values of the functions ω_k (2.8) that enter Eqs. (2.5), and the system of transcendental equations (2.5) closed relative to α_k ($k = 1$ and 2) may be transformed to

$$\cos(2\varphi_2 - \psi) = (b^2 + c^2 - \sigma_1^2\omega_{01}^2 + \sigma_2^2\omega_{02}^2)(2\sigma_2\omega_{02}\sqrt{b^2 + c^2})^{-1},$$
$$\tan 2\varphi_1 = (2p_\tau - \sigma_2\omega_{02}\sin 2\varphi_2)\Big(2p_n - \sum_k \sigma_k\omega_{0k} - \sigma_2\omega_{02}\cos 2\varphi_2\Big)^{-1}, \tag{2.11}$$
$$\alpha_k(\Gamma_\omega) = \alpha_{0k}(s) = \varphi_k + \beta \quad (k = 1,\ 2), \quad \cos\psi = c/\sqrt{b^2 + c^2}, \quad \sin\psi = b/\sqrt{b^2 + c^2},$$

where $b = 2p_\tau$ and $c = 2p_n - \sigma_1\omega_{01} - \sigma_2\omega_{02}$.

Thus, Eqs. (2.9) and (2.10) and the boundary conditions (2.8) and (2.11) define the expressions for the RR parameters α_k and ω_k in an analytical form in the absence of volume loads. The first equation of (2.9) in the plane x_1Ox_2 defines a straight line passing at an angle α_k to the Ox_1 axis. Hence, in the absence of volume loads, the reinforcement trajectories are straight lines in the asymptotic approximation.

The function β in (2.11) defines the direction of the external normal to the contour Γ_p. For the fibers to enter the construction on this contour, the solution of system (2.11) should be sought in the open intervals $\varphi_k \in (\pi/2, 3\pi/2)$. In these intervals, the first equation in (2.11) may have up to two different roots depending on ψ and the value of the right part; the second equation has only one root for known φ_2. Thus, system (2.11) may have up to two different sets of solutions relative to $\alpha_k(\Gamma_p) = \alpha_{0k}(s)$; hence, Eqs. (2.9) and (2.10) with allowance for (2.8) determine two sets of RR parameters, which satisfy the same problem of rational design. It follows from the asymptotic analysis of the system of resolving equations that the RR problem may have two solutions.

Note that Eqs. (2.9) and (2.10) are the solutions of the Cauchy problem for system (2.1) ($b_i = 0$) under the initial conditions $\alpha_k(\Gamma_\omega) = \alpha_{0k}(s)$ and $\omega_k(\Gamma_\omega) = \omega_{0k}(s)$. In the presence of volume loads ($b_i \neq 0$), the Cauchy problem with the initial conditions (2.8) and (2.11) (assuming that all input data are analytical) is posed for system (2.1) (owing to the fact that all its characteristics are real). As in the case of the absence of volume forces, we can obtain two sets of RR parameters that satisfy the same initial problem.

If the Cauchy problem for system (2.1), (2.2) with the initial conditions (2.8) and (2.11) is integrated in the entire region G occupied by the construction in the plane, i.e., the functions α_k and ω_k are known everywhere throughout G, then, as is shown in [1, 2], if conditions (1.14) are satisfied, Eq. (2.3) is a second-order linear elliptic equations relative to the temperature θ, which corresponds to the linear thermal boundary conditions (2.7). The boundary problems for second-order linear elliptic equations are well studied in [7].

If the functions α_k and θ ($k = 1,\ 2$) are known from subsystems (2.1)–(2.3), then subsystem (2.4) is a first-order linear hyperbolic system relative to the displacements u_k, and its characteristics are determined by the angles α_k and coincide with the characteristics of system (2.1), (2.2). The initial conditions for subsystem (2.4) are the kinematic boundary conditions (2.6) set on the part of the contour Γ_u, where the fibers are assumed to go out of the construction. In the absence of volume loads and a for constant temperature in the construction ($b_i = 0$ and $\theta = \text{const}$), the solution of the Cauchy problem (2.4), (2.6) may be obtained analytically. Indeed, it was shown previously that, for $b_i = 0$ ($i = 1,\ 2$), the characteristics of system (2.1), (2.2), and hence, those of system (2.4) are rectilinear, i.e., $\partial_s(\alpha_k, \alpha_k) = 0$ ($k = 1$ and 2). Therefore, system (2.4) for $\theta = \text{const}$ may be written in the form of Riemann invariants [6] with a zero right part; then the solution of the Cauchy problem (2.4), (2.6) can be easily constructed:

$$u_1 \cos\alpha_k + u_2 \sin\alpha_k - (\varepsilon_k + \alpha_{ak}\theta)(x_1 \cos\alpha_k + x_2 \sin\alpha_k) = u_{10}(s)\cos\alpha_{ku}(s)$$

$$+ u_{20}(s)\sin\alpha_{ku}(s) - (\varepsilon_k + \alpha_{ak}\theta)[\eta_1(s)\cos\alpha_{ku}(s) + \eta_2(s)\sin\alpha_{ku}(s)] \qquad (k = 1,\ 2). \tag{2.12}$$

Here $\varepsilon_k + \alpha_{ak}\theta = \text{const}$, $\alpha_{ku}(s)$ are the initial values of the functions α_k on the contour Γ_u, which are known from the solution of the Cauchy problem (2.8)–(2.11), and η_k are the functions defining the equation for Γ_u [$x_k = \eta_k(s)$, where $k = 1$ and 2]. [Since the characteristics of system (2.4) are rectilinear for $b_i = 0$, the equalities $\alpha_k(x_1, x_2) = \alpha_{ku}(s)$ are valid in Eqs. (2.12).]

Thus, in the asymptotic approximation ($\lambda \to 0$), the solution of the thermoelastic RR problem reduced to sequential integration of the Cauchy problem (2.1), (2.2), (2.8), (2.11), the boundary-value problem (2.3), (2.7), and the Cauchy problem (2.4), (2.6). The reason for this nonclassical reduction of the boundary-value problem of RR as $\lambda \to 0$ to the Cauchy problem is that the system of resolving equations (1.1)–(1.4) and the boundary conditions (1.9)–(1.11) and (1.13) form a system of nonlinear equations with a singular perturbation [8].

It should be noted that Eqs. (2.1), (2.2), and (2.11) [in particular, Eqs. (2.9)–(2.11)] do not contain the temperature θ. Therefore, in the asymptotic approximation (i.e., within the framework of the fibrous model), the temperature affects only the construction compliance [see (2.4) and (2.12)] but has no influence on the RR structure. To take this effect into account, it is necessary to construct higher-order approximations using the method of the small parameter [8] or the following iterative process. Let $\alpha_k^{(m)}$, $\omega_k^{(m)}$, $u_k^{(m)}$, and $\theta^{(m)}$ ($k = 1$ and 2) be known mth approximations of unknown functions, then, the $(m+1)$th approximation for them may be obtained by integrating the equations

$$A_i(\boldsymbol{\omega}^{(r)}, \boldsymbol{\alpha}^{(r)}) = -b_i(\boldsymbol{\omega}^{(r)}) - \lambda B_i(\boldsymbol{\omega}^{(m)}, \boldsymbol{u}^{(m)}, \theta^{(m)}) \qquad (i = 1,\ 2); \tag{2.13}$$

$$\partial_s(\alpha_k^{(r)}, \omega_k^{(r)}) + \omega_k^{(r)} \partial_n(\alpha_k^{(r)}, \alpha_k^{(r)}) = 0 \qquad (k = 1,\ 2); \tag{2.14}$$

$$(\Lambda_{11}^{(r)}\theta_{,1}^{(r)} + \Lambda_{12}^{(r)}\theta_{,2}^{(r)})_{,1} + (\Lambda_{21}^{(r)}\theta_{,1}^{(r)} + \Lambda_{22}^{(r)}\theta_{,2}^{(r)})_{,2} + \mu(\theta_* - \theta^{(r)}) = -Q(\boldsymbol{\omega}^{(r)}); \tag{2.15}$$

$$\partial_s(\alpha_k^{(r)}, u_1^{(r)}) \cos\alpha_k^{(r)} + \partial_s(\alpha_k^{(r)}, u_2^{(r)}) \sin\alpha_k^{(r)} = \varepsilon_k + \alpha_{ak}\theta^{(r)} \quad (k = 1,\ 2, \quad r = m + 1) \tag{2.16}$$

with the boundary conditions

$$C_n(\boldsymbol{\alpha}^{(r)}, \boldsymbol{\omega}^{(r)}) = p_n - \lambda D_n(\boldsymbol{\omega}^{(m)}, \boldsymbol{u}^{(m)}, \theta^{(m)}),$$
$$C_\tau(\boldsymbol{\alpha}^{(r)}, \boldsymbol{\omega}^{(r)}) = 2p_\tau - \lambda D_\tau(\boldsymbol{\omega}^{(m)}, \boldsymbol{u}^{(m)}); \tag{2.17}$$

$$u_i^{(r)}(\Gamma_u) = u_{i0}(s) \qquad (i = 1,\ 2); \tag{2.18}$$

$$\chi_0[(\Lambda_{11}^{(r)}\theta_{,1}^{(r)} + \Lambda_{12}^{(r)}\theta_{,2}^{(r)})n_1 + (\Lambda_{21}^{(r)}\theta_{,1}^{(r)} + \Lambda_{22}^{(r)}\theta_{,2}^{(r)})n_2 + q_0] + \chi_1(\theta^{(r)} - \theta_0) = 0; \tag{2.19}$$

$$\omega_k^{(r)}(\Gamma_\omega) = \omega_{0k}(s) \qquad (k = 1,\ 2, \qquad r = m + 1). \tag{2.20}$$

For the beginning of the iterative process, we use the zero approximation in the form

$$u_i^{(0)} = 0, \qquad \theta^{(0)} = 0 \qquad (i = 1,\ 2). \tag{2.21}$$

Here $\boldsymbol{\alpha}^{(r)}$, $\boldsymbol{\omega}^{(r)}$, and $\boldsymbol{u}^{(r)}$ are vector functions similar to those in (1.7); the expressions for $\Lambda_{ij}^{(r)}$ are obtained from (1.8) by substituting α_k and ω_k by their rth approximations.

A comparison of equations and boundary conditions of the iterative process (2.13)–(2.20) with the corresponding equations and boundary conditions (2.1)–(2.8) shows that they differ only by the perturbed right parts in Eqs. (2.13) and (2.17); at the first iteration ($r = 1$), due to the initial approximation (2.21), Eqs. (2.1)–(2.8) coincide with Eqs. (2.13)–(2.20). Therefore, all the results obtained above in analyzing Eqs. (2.1)–(2.8) are valid for the iterative-process equations (2.13)–(2.20). In particular, the iterative process allows one to obtain two solutions of the RR problem; in the absence of volume loads ($b_i = 0$), the RR parameters and displacements (for $\theta = \text{const}$) in the first approximation are determined by equalities (2.9)–(2.12). The right parts in Eqs. (2.13) and (2.17) [in contrast to (2.1) and (2.5)] depend on the temperature $\theta^{(m)}$ calculated at the mth step of the iterative process. Hence, the RR structure determined by the iterative process depends on the thermal action on the construction.

To verify the convergence of the iterative process, it is reasonable to substitute the rth approximations of the unknown functions into the system of resolving equations (1.1)–(1.4) and the corresponding boundary conditions (1.9)–(1.11) and (1.13) and determine the arising residuals. The residuals arise only in Eqs. (1.1) and boundary conditions (1.9) and have the following form:

$$\delta_i^{(r)} = \lambda[B_i(\boldsymbol{\omega}^{(r)}, \boldsymbol{u}^{(r)}, \theta^{(r)}) - B_i(\boldsymbol{\omega}^{(m)}, \boldsymbol{u}^{(m)}, \theta^{(m)})] \qquad (i = 1,\ 2),$$
$$\delta_n^{(r)} = \lambda[D_n(\boldsymbol{\omega}^{(r)}, \boldsymbol{u}^{(r)}, \theta^{(r)}) - D_n(\boldsymbol{\omega}^{(m)}, \boldsymbol{u}^{(m)}, \theta^{(m)})], \tag{2.22}$$
$$\delta_\tau^{(r)} = \lambda[D_\tau(\boldsymbol{\omega}^{(r)}, \boldsymbol{u}^{(r)}) - D_\tau(\boldsymbol{\omega}^{(m)}, \boldsymbol{u}^{(m)})] \qquad (r = m + 1).$$

3. Analysis of Some Solutions of the Thermoelastic RR Problem. We consider an extended rectangular plate of unit length, which is aligned along the Ox_2 axis. Assuming that the load, thermal action, attachment, and reinforcement of the construction do not vary in the longitudinal direction and the local butt-end effects may be neglected, the solution of the RR problem depends only on the variable x_1. We assume that the plate is uniformly heated or cooled ($\theta = \text{const}$, $Q = 0$, and $\mu = 0$); then, integrating Eqs. (1.1) and (1.2), we obtain the following system of resolving equations that describe the thermoelastic RR problem:

$$\lambda a_1\Big(1 - \sum_k \omega_k\Big)(\varepsilon_{11} - \alpha_c\theta/a_2) + \sum_k \sigma_k\omega_k\cos^2\alpha_k = P_1(x_1),$$
$$\lambda a_2\Big(1 - \sum_k \omega_k\Big)\varepsilon_{12} + \sum_k \sigma_k\omega_k\sin\alpha_k\cos\alpha_k = P_2(x_1); \tag{3.1}$$

$$\omega_k\cos\alpha_k = \omega_{k^*} = \text{const} \qquad (k = 1,\ 2); \tag{3.2}$$

$$\varepsilon_{11}\cos^2\alpha_k + \varepsilon_{12}\sin 2\alpha_k = \varepsilon_k + \alpha_{ak}\theta = \text{const} \qquad (k = 1,\ 2). \tag{3.3}$$

TABLE 1

Function	Elastic problem ($\theta = 0$)			Thermoelastic problem ($\theta = 4$)		
	$\lambda = 0$	$\lambda = 0.01$	$\lambda = 0.1$	$\lambda = 0$	$\lambda = 0.01$	$\lambda = 0.1$
α_1	0.615 48	0.627 29	0.757 50	0.615 48	0.590 61	0.251 00
ε_{11}	1.5000	1.5256	1.8943	2.1000	2.0293	1.4920

Here $P_i(x_1) = p_i - \int\limits_0^{x_1} b_i(s)\,ds$ $(i = 1,\ 2)$, $p_1 = p_n$, $p_2 = p_\tau$, and ω_{k^*} are integration constants, which have the meaning of the total cross-sectional area of reinforcement of the kth family intersecting the area of unit length (along x_2) and orthogonal to the direction x_1 [ω_{k^*} may be used instead of ω_{0k} in (1.13)], static boundary conditions (1.9) are set at the edge $x_1 = 0$, and the plate is rigidly fixed at the edge $x_1 = 1$. The heat-conduction equation (1.4) for $\theta = \text{const}$, $Q = 0$, and $\mu = 0$ is identically satisfied; therefore, it is not considered. In addition, it is assumed for simplicity that the reduced load b_i is independent of the reinforcement intensity ω_k, which is valid for $b_i = F_{ci} = F_{ki}$ $(i, k = 1, 2)$ [see (1.1)]. These equalities are satisfied under the action of distributed surface loads that arise, for instance, in a viscous liquid flow past a plate.

System (3.1)–(3.3) may be transformed eliminating the functions ω_k, ε_{11}, and ε_{12} from (3.1) taking into account Eqs. (3.2) and (3.3). As a result, we obtain a system of transcendental equations that define two unknowns $\alpha_k(x_1)$ ($k = 1$ and 2) at each point x_1.

We assume that there is no load in the longitudinal direction x_2 ($p_\tau = 0$ and $b_2 = 0$). In this case, it is reasonable to introduce into the construction the fibers of two families made of the same material ($\sigma_2 = \sigma_1$, $E_2 = E_1$, and $\alpha_{a2} = \alpha_{a1}$) and applied with an identical intensity ($\omega_2 = \omega_1$) symmetrically with respect to the direction x_1 ($\alpha_2 = -\alpha_1$). This choice of the reinforcement structure allows us to eliminate shear strains ($\varepsilon_{12} = 0$) from consideration. In doing so, the second equation in (3.1) is satisfied identically, and we can eliminate ω_k and ε_{11} from the first one using (3.2) and (3.3):

$$\lambda a_1(1 - 2\omega_{1^*}/g)[(\varepsilon_1 + \alpha_{a1}\theta)/g^2 - \alpha_c\theta/a_2] + 2\sigma_1\omega_{1^*}g = P_1(x_1), \quad g = \cos\alpha_1(x_1). \tag{3.4}$$

By means of multiplication by g^3, Eq. (3.4) is reduced to a fourth-order algebraic equation with respect to g with coefficients depending on θ and ω_{1^*}. Consequently, Eq. (3.4) and, hence, the thermoelastic RR problem considered may have up to four solutions; each solution depends parametrically on the level of the thermal action θ and the amount of reinforcement introduced into the construction (on ω_{1^*}).

If we consider Eq. (3.4) on the edge $x_1 = 0$ and set the boundary condition (1.13) instead of ω_{1^*}, then Eq. (3.4) acquires the following form:

$$\lambda a_1(1 - 2\omega_{01})[(\varepsilon_1 + \alpha_{a1}\theta)/g^2 - \alpha_c\theta/a_2] + 2\sigma_1\omega_{01}g^2 = p_n. \tag{3.5}$$

Equation (3.5) defined the RR structure at all points of the plate if $b_1(x_1) = 0$. By means of multiplication by g^2, Eq. (3.5) is reduced to a biquadratic equation with respect to g. Obviously, among all the real roots of Eqs. (3.4) and (3.5), the solutions of the RR problem are only the roots that satisfy the inequalities $0 \leqslant g = \cos\alpha_1 \leqslant 1$ (the negative values of g are eliminated from consideration by virtue of the assumption that the fibers enter the plate at the edge $x_1 = 0$) and the physical constraints (1.14).

Table 1 shows the values of unknown functions determining the solution of the elastic ($\theta = 0$) and thermoelastic ($\theta = 4$) problems of rational reinforcement of a rectangular plate for different λ and the following input data: $\nu = 0.3$, $E_1 = \sigma_1 = 1$, $b_1 = b_2 = 0$, $p_n = 0.4$, $p_\tau = 0$, $\alpha_c = 1$, $\alpha_{a1} = 0.1$, and $\omega_{01} = 0.3$. The values in Table 1 are obtained by solving Eq. (3.5). The second solution of the RR problem, which corresponds to the second root of Eq. (3.5), has a singularity of the type $\alpha_1 \to \pi/2$, $|\varepsilon_{11}| \to \infty$ as $\lambda \to 0$ and is not considered in the present paper.

It follows from Table 1 that, if high-modulus reinforcement is used ($\lambda \approx 0.01$), the solutions of the elastic and thermoelastic RR problems differ from the solutions in the asymptotic approximation ($\lambda \to 0$) by no more than 5%. Therefore, in the case of high-modulus reinforcement, one can use the solution constructed

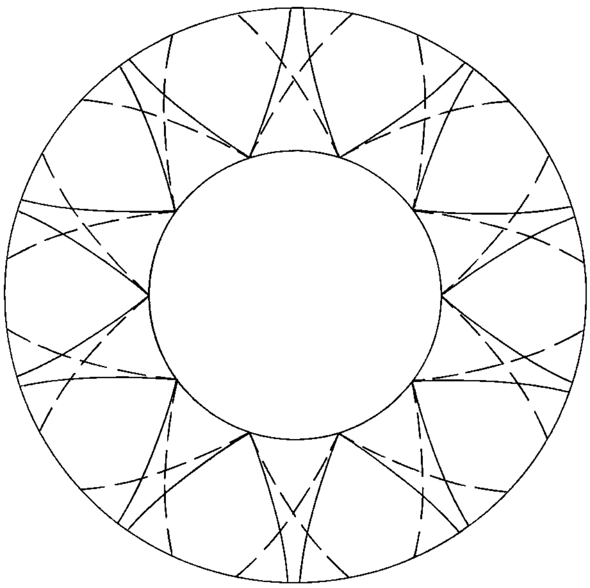

Fig. 1

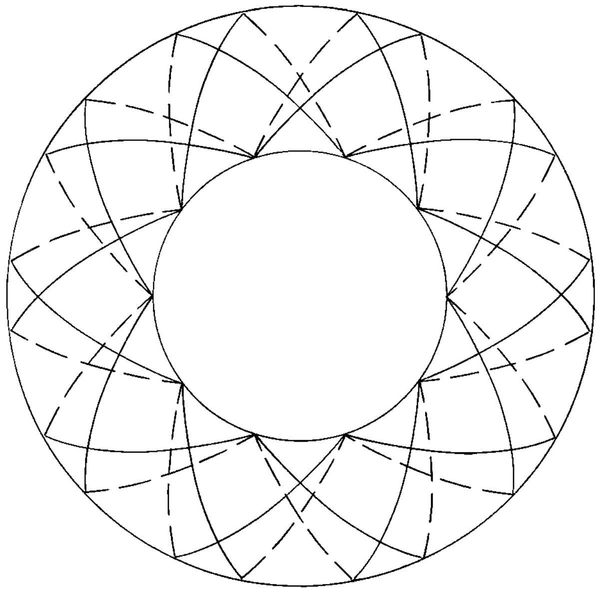

Fig. 2

on the basis of the fibrous model of mechanical behavior of the reinforced layer ($\lambda = 0$) as an approximate solution of the RR problem.

In addition, as was noted in Sec. 2 and as follows from Table 1, the RR structures in the elastic and thermoelastic problems coincide in the asymptotic approximation; therefore, if it is necessary to determine only the RR structure (without determining the strain of the construction and the temperature field in it) for $\lambda \approx 0.01$, it is sufficient to solve the Cauchy problem (2.1), (2.2), (2.8), (2.11). Hence, in this case, the effect of the thermal action on the RR structure may be ignored.

We consider examples of rational reinforcement of annular plates with uniform cooling and heating. The annular plate is bounded by two circles of radii r_0 and r_1 ($r_0/r_1 = 0.5$). A uniform pressure $p_n = 0.5$ ($p_\tau = 0$) is applied to the internal contour Γ_p, and the plate is rigidly fixed at the external contour Γ_u. There are no distributed loads; the construction is uniformly cooled: $\theta = -2$. Owing to the symmetry of the construction and the absence of circumferential loads, it is reasonable to introduce into the plate two families of reinforcement made of the same material ($\sigma_1 = \sigma_2 = E_1 = E_2 = 1$, $\alpha_{a1} = \alpha_{a2} = 1.5$, $\lambda = 0.1$, $\nu = 0.25$, and $\alpha_c = 1$) and applied with radial symmetry and identical intensity [$\omega_2(r) = \omega_1(r)$]. This allows us to eliminate shear strains ($\varepsilon_{r\varphi} = 0$). The solid curves in Fig. 1 show the RR structure corresponding to this thermoelastic problem under the boundary conditions $\omega_k(\Gamma_p) = 0.4$ ($k = 1$ and 2); the dashed curves show the RR structure of the construction that does not experience the thermal action. A comparison of the reinforcement projects in Fig. 1 shows that plate cooling leads to condensation of reinforcement trajectories [the number of fibers of the kth family entering the construction on the contour section Γ_p of unit length is determined by the product $\omega_k(\Gamma_p)\cos(\alpha_k(\Gamma_p) - \beta)$ [3] or by the value of ω_{k^*} in (3.2)], but the length of fibers decreases thereby, and the total consumption of the fibrous material decreases by 4%. In the case of a more intense cooling of the plate at the external contour, the fibers of different families contact each other ($\alpha_1 = \alpha_2$), which leads to the emergence of cuspidal points of reinforcement trajectories.

The solid curves in Fig. 2 show the project of rational reinforcement obtained for the above-described construction, which is uniformly heated to a temperature $\theta = 1.2$. Plate heating involves a significant change in the RR structure as compared to the structure of the elastic project (dashed curves); the curvature of reinforcement trajectories increases and retains the same sign as in the elastic case. The curvatures of reinforcement trajectories have different signs in the case of plate cooling and in the elastic project (see Fig. 1). A more intense heating of the plate increases the absolute value of the curvature of reinforcement trajectories, and the second condition in (1.14) is no longer satisfied after a certain limiting value of temperature on the external contour, i.e., in such a project, the fibers will bulge out of reinforcement planes.

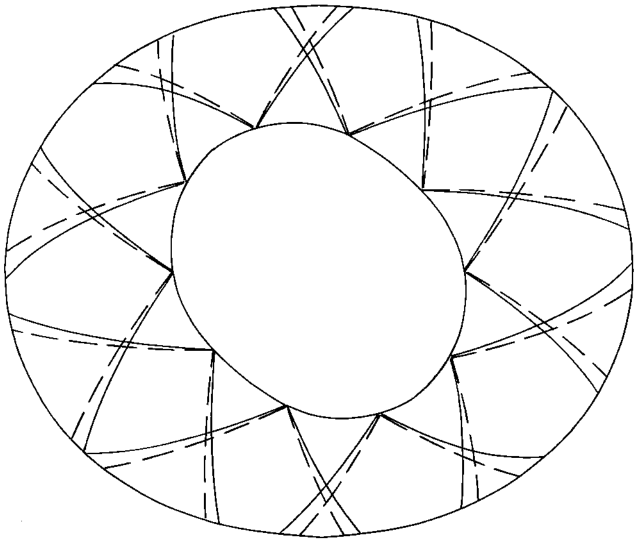

Fig. 3

The RR trajectories shown in Figs. 1 and 2, the results of Table 1, and the series of calculations performed by the authors show that, if the coefficients of linear thermal expansion of the fibers are greater than the corresponding coefficient of the binder, then the RR trajectories become more scarce if the sign of temperature coincides with the sign of stress in the fibers; otherwise, the RR trajectories become more dense. If these coefficients of the fibers are lower than the coefficient of the binder (see Table 1), then, the RR trajectories become more dense is the signs of the temperature and stress in the fibers coincide; otherwise, the RR trajectories become more scarce.

We consider an asymmetric thermoelastic RR problem. Let a plane construction be bounded by two contours Γ_p and Γ_u defined in a polar coordinate system by the equations $r(\varphi) = 0.5 - 0.05\sin 2\varphi$ and $r(\varphi) = 1 + 0.08\cos 2\varphi$, respectively. A uniform normal load $p_n = 0.5$ ($p_\tau = 0$) is applied to the internal contour Γ_p, and the construction is rigidly fixed at the external contour Γ_u. There are no volume loads, and the plate is uniformly heated to a temperature $\theta = 2$. The mechanical characteristics of the phases of the composition are the same as in the above examples, except for $\lambda = 0.05$. The boundary conditions for reinforcement intensities are set on the internal contour: $\omega_{0k}(\varphi) = 0.4$ ($k = 1$ and 2). The solid curves in Fig. 3 show the RR structure corresponding to this asymmetric problem and obtained by the iterative process (2.13)–(2.21); the dashed curves show the structure of the elastic project ($\theta = 0$).

Since we have $\lambda \approx 0.01$ in the example considered, the RR structure in the thermoelastic case differs insignificantly from the RR structure of the elastic project, which could be expected. Thus, in determining the RR structure in the case of high-modulus reinforcement ($\lambda \approx 0.01$), the thermal action may be ignored, since the thermoelastic RR project differs insignificantly from the corresponding elastic project. Nevertheless, even for $\lambda \approx 0.01$, the temperature has a significant effect on the deformability of the construction and on the stress–strained state in the binder. In the last example (see Fig. 3), the maximum value of the reinforcement intensity in the binder in the heated construction is 2.41 times greater than in the nonheated construction ($\theta = 0$). These results are supported by the data listed in Table 1.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 99-01-00549).

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